13.5

Calculating Gradients

In Exercises 1–6, find the gradient of the function at the given point. Then sketch the gradient, together with the level curve that passes through the point.

1. $f(x,y) = y - x$, $(2,1)$ **2.** $f(x,y) = \ln(x^2 + y^2)$, $(1,1)$

3. $g(x,y) = xy^2$, $(2,-1)$ **4.** $g(x,y) = \frac{x^2}{2} - \frac{y^2}{2}$, $(\sqrt{2},1)$

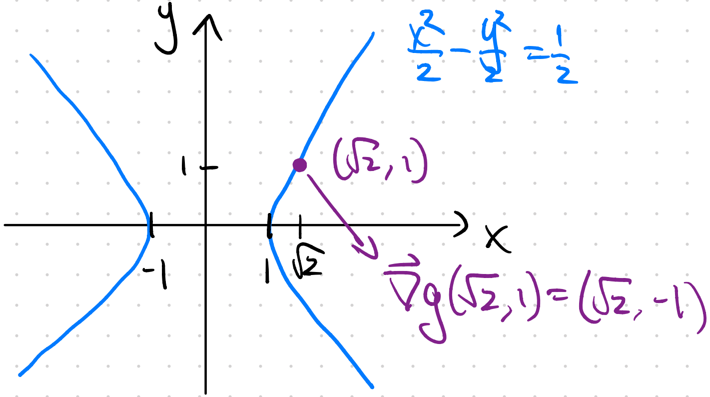
Sol'n: $\vec{\nabla} g(\sqrt{2},1) = \left(\frac{\partial g}{\partial x}\Big|_{(\sqrt{2},1)}, \frac{\partial g}{\partial y}\Big|_{(\sqrt{2},1)} \right)$

$\frac{\partial g}{\partial x}\Big|_{(\sqrt{2},1)} = x\Big|_{(\sqrt{2},1)} = \sqrt{2}$. $\frac{\partial g}{\partial y}\Big|_{(\sqrt{2},1)} = -y\Big|_{(\sqrt{2},1)} = -1$.

So $\vec{\nabla} g(\sqrt{2},1) = (\sqrt{2}, -1)$

$g(\sqrt{2},1) = \frac{(\sqrt{2})^2}{2} - \frac{(1)^2}{2} = 1 - \frac{1}{2} = \frac{1}{2}$.

So level curve given by $\frac{x^2}{2} - \frac{y^2}{2} = \frac{1}{2}$.


$\frac{x^2}{2} - \frac{y^2}{2} = \frac{1}{2}$
y
x
1
-1
1
$\sqrt{2}$
$(\sqrt{2}, 1)$
$\vec{\nabla} g(\sqrt{2}, 1) = (\sqrt{2}, -1)$

13.5 In Exercises 7–10, find ∇f at the given point.

10. $f(x,y,z) = e^{x+y}\cos z + (y+1)\arcsin x, \quad (0,0,\pi/6)$

Sol'n:

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0,\frac{\pi}{6})} = \left.\frac{\partial}{\partial x}\left(e^{x+y}\cos z + (y+1)\arcsin x\right)\right|_{(0,0,\frac{\pi}{6})}$$

$$= \left.\left(e^{x+y}\cos z + \frac{y+1}{\sqrt{1-x^2}}\right)\right|_{(0,0,\frac{\pi}{6})}$$

$$= e^{0+0}\cos\left(\frac{\pi}{6}\right) + \frac{0+1}{\sqrt{1-0^2}}$$

$$= \frac{\sqrt{3}}{2} + 1$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0,\frac{\pi}{6})} = \left.\frac{\partial}{\partial y}\left(e^{x+y}\cos z + (y+1)\arcsin x\right)\right|_{(0,0,\frac{\pi}{6})}$$

$$= \left.\left(e^{x+y}\cos z + \arcsin x\right)\right|_{(0,0,\frac{\pi}{6})}$$

$$= e^{0+0}\cos\left(\frac{\pi}{6}\right) + \arcsin(0)$$

$$= \frac{\sqrt{3}}{2}$$

$$\left.\frac{\partial f}{\partial z}\right|_{(0,0,\frac{\pi}{6})} = \left.\frac{\partial}{\partial z}\left(e^{x+y}\cos z + (y+1)\arcsin x\right)\right|_{(0,0,\frac{\pi}{6})}$$

$$= \left.\left(-e^{x+y}\sin z\right)\right|_{(0,0,\frac{\pi}{6})}$$

$$= -e^{0+0}\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{-1}{2}.$$

So $\vec{\nabla} f(0,0,\frac{\pi}{6}) = \left(\frac{\sqrt{3}}{2}+1, \frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

13.5

Finding Directional Derivatives

In Exercises 11–18, find the derivative of the function at P_0 in the direction of $\mathbf{v}$.

11. $f(x, y) = 2xy - 3y^2$, $P_0(5, 5)$, $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$

12. $f(x, y) = 2x^2 + y^2$, $P_0(-1, 1)$, $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

Sol'n: $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$. $\|\vec{v}\| = \sqrt{3^2 + (-4)^2} = 5$.

So $\vec{u} = \frac{1}{5}(3, -4)$.

We'll use $D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u}$

$\left.\dfrac{\partial f}{\partial x}\right|_{(-1,1)} = \left.4x\right|_{(-1,1)} = -4$. $\left.\dfrac{\partial f}{\partial y}\right|_{(-1,1)} = \left.2y\right|_{(-1,1)} = 2$.

So $\vec{\nabla} f(P_0) = (-4, 2)$ and

$D_{\vec{v}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{v} = (-4, 2) \cdot \frac{1}{5}(3, -4) = \boxed{-4}$

13.5 18. $h(x, y, z) = \cos xy + e^{yz} + \ln zx$, $P_0(1, 0, 1/2)$,
$\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

Sol'n: Let $u = \frac{\vec{v}}{\|\vec{v}\|}$. $\|\vec{v}\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$. So $\vec{u} = \frac{1}{3}(1, 2, 2)$

$$D_{\vec{u}} h(P_0) = \vec{\nabla} h(P_0) \cdot \vec{u}.$$

$$\left.\frac{\partial h}{\partial x}\right|_{(1,0,\frac{1}{2})} = \left.\left(-y\sin(xy) + \frac{1}{x}\right)\right|_{(1,0,\frac{1}{2})} = 1$$

$$\left.\frac{\partial h}{\partial y}\right|_{(1,0,\frac{1}{2})} = \left.\left(-x\sin(xy) + ze^{yz}\right)\right|_{(1,0,\frac{1}{2})} = \frac{1}{2}$$

$$\left.\frac{\partial h}{\partial z}\right|_{(1,0,\frac{1}{2})} = \left.\left(ye^{yz} + \frac{1}{z}\right)\right|_{(1,0,\frac{1}{2})} = 2.$$

So $D_{\vec{u}} h(P_0) = (1, \frac{1}{2}, 2) \cdot \frac{1}{3}(1, 2, 2) = 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} + 2 \cdot \frac{2}{3} = \boxed{2}$

13.5

In Exercises 19–24, find the directions in which the functions increase most rapidly, and the directions in which they decrease most rapidly, at P_0. Then find the derivatives of the functions in these directions.

20. $f(x, y) = x^2y + e^{xy}\sin y, \quad P_0(1, 0)$

Sol'n: We have $-\|\vec{\nabla} f(P_0)\| \leq D_{\vec{v}} f(P_0) \leq \|\vec{\nabla} f(P_0)\|$

with equality when $\vec{v} = \pm \dfrac{\vec{\nabla} f(P_0)}{\|\vec{\nabla} f(P_0)\|}$.

So we compute $\vec{\nabla} f(P_0)$:

$$\left.\frac{\partial f}{\partial x}\right|_{(1,0)} = \left.\left(2xy + ye^{xy}\sin y\right)\right|_{(1,0)} = 0.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,0)} = \left.\left(x^2 + xe^{xy}\sin y + e^{xy}\cos y\right)\right|_{(1,0)} = 1 + 1 = 2$$

So $\vec{\nabla} f(P_0) = (0, 2)$. $\quad \|\vec{\nabla} f(P_0)\| = \sqrt{0^2 + 2^2} = 2.$

So $D_{\vec{v}}f(P_0)$ is largest when $\vec{v} = \frac{1}{2}(0,2) = (0,1)$

with value $D_{\vec{v}}f(P_0) = \vec{\nabla}f(P_0) \cdot \vec{v} = (0,2) \cdot (0,1) = 2$.

and $D_{\vec{v}}f(P_0)$ is smallest when $\vec{v} = \frac{1}{2}(0,-2) = (0,-1)$

with value $D_{\vec{v}}f(P_0) = \vec{\nabla}f(P_0) \cdot \vec{v} = (0,2) \cdot (0,-1) = -2$

13.5

30. Let $f(x, y) = \frac{(x-y)}{(x+y)}$. Find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right)$ for which

a. $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right)$ is largest

b. $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right)$ is smallest

c. $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = 0$

d. $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = -2$

e. $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = 1$

Sol'n: $D_{\vec{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \vec{\nabla}f\left(-\frac{1}{2}, \frac{3}{2}\right) \cdot \vec{u}$.

$$\left.\frac{\partial f}{\partial x}\right|_{\left(-\frac{1}{2}, \frac{3}{2}\right)} = \left.\frac{(1)(x+y) - (x-y)(1)}{(x+y)^2}\right|_{\left(-\frac{1}{2}, \frac{3}{2}\right)} = \left.\frac{x+y-x+y}{(x+y)^2}\right|_{\left(-\frac{1}{2}, \frac{3}{2}\right)} = \left.\frac{2y}{(x+y)^2}\right|_{\left(-\frac{1}{2}, \frac{3}{2}\right)} = 3$$

$$\left.\frac{\partial f}{\partial y}\right|_{\left(-\frac{1}{2}, \frac{3}{2}\right)} = \left.\frac{(-1)(x+y) - (x-y)(1)}{(x+y)^2}\right|_{\left(-\frac{1}{2}, \frac{3}{2}\right)} = \left.\frac{-x-y-x+y}{(x+y)^2}\right|_{\left(-\frac{1}{2}, \frac{3}{2}\right)} = \left.\frac{-2x}{(x+y)^2}\right|_{\left(-\frac{1}{2}, \frac{3}{2}\right)} = 1$$

So $\vec{\nabla}f\left(-\frac{1}{2}, \frac{3}{2}\right) = (3, 1)$ and $\left\|\vec{\nabla}f\left(-\frac{1}{2}, \frac{3}{2}\right)\right\| = \sqrt{3^2+1^2} = \sqrt{10}$

a) So $D_{\vec{u}}f(-\frac{1}{2},\frac{3}{2})$ is largest when $\vec{u}=\frac{1}{\sqrt{10}}(3,1)$.

b) $D_{\vec{u}}f(\frac{-1}{2},\frac{3}{2})$ is smallest when $\vec{u}=\frac{-1}{\sqrt{10}}(3,1)$.

c) Take $\vec{u}=(u_1,u_2)$. Then

$$D_{\vec{u}}f(\tfrac{-1}{2},\tfrac{3}{2}) = \vec{\nabla}f(\tfrac{-1}{2},\tfrac{3}{2})\cdot\vec{u} = (3,1)\cdot(u_1,u_2) = 3u_1+u_2.$$

Have $\begin{cases} 0=3u_1+u_2. \\ 1=\sqrt{u_1^2+u_2^2} \end{cases}$ $\Rightarrow u_2=-3u_1$

$1=\sqrt{u_1^2+(-3u_1)^2} \Rightarrow 1=u_1^2+9u_1^2=10u_1^2. \Rightarrow u_1^2=\frac{1}{10}$

$\Rightarrow u_1=\pm\frac{1}{\sqrt{10}}.$

$\Rightarrow u_2=\mp\frac{3}{\sqrt{10}}.$

So $\vec{u}=\left(\frac{1}{\sqrt{10}},\frac{-3}{\sqrt{10}}\right)$ or $\vec{u}=\left(-\frac{1}{\sqrt{10}},\frac{3}{\sqrt{10}}\right)$.

d) Have $\begin{cases} -2=3u_1+u_2 \\ 1=\sqrt{u_1^2+u_2^2} \end{cases}$ $\Rightarrow u_2=-2-3u_1.$

So $1 = u_1^2 + (-2-3u_1)^2 = u_1^2 + 4 + 12u_1 + 9u_1^2$

$\Rightarrow 10u_1^2 + 12u_1 + 3 = 0. \Rightarrow u_1 = \dfrac{-6 \pm \sqrt{6}}{10}.$

When $u_1 = \dfrac{-6-\sqrt{6}}{10}$, $u_2 = -2-3\left(\dfrac{-6-\sqrt{6}}{10}\right) = \dfrac{-2+3\sqrt{6}}{10}$

$u_1 = \dfrac{-6+\sqrt{6}}{10}$, $u_2 = -2-3\left(\dfrac{-6+\sqrt{6}}{10}\right) = \dfrac{-2-3\sqrt{6}}{10}$

So $\vec{u} = \left(\dfrac{-6-\sqrt{6}}{10}, \dfrac{-2+3\sqrt{6}}{10}\right)$ or $\vec{u} = \left(\dfrac{-6+\sqrt{6}}{10}, \dfrac{-2-3\sqrt{6}}{10}\right)$ //

e) $\begin{cases} 1 = 3u_1 + u_2 \\ 1 = \sqrt{u_1^2 + u_2^2} \end{cases}$ $\Rightarrow u_2 = 1 - 3u_1$

So $1 = u_1^2 + (1-3u_1)^2 = u_1^2 + 1 - 6u_1 + 9u_1^2$

$\Rightarrow 10u_1^2 - 6u_1 = 0. \Rightarrow u_1(10u_1 - 6) = 0.$

So $u_1 = 0$ or $\frac{3}{5}$

When $u_1 = 0$, $u_2 = 1$. When $u_1 = \frac{3}{5}$, $u_2 = 1 - 3\left(\frac{3}{5}\right) = \frac{-4}{5}$

So $\vec{u} = (0, 1)$ or $\vec{u} = \left(\frac{3}{5}, \frac{-4}{5}\right)$.

13.6 **22.** By about how much will

$$f(x, y, z) = e^x \cos yz$$

change as the point $P(x, y, z)$ moves from the origin a distance of $ds = 0.1$ unit in the direction of $2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$?

Sol'n:

By textbook, $df = \left(\vec{\nabla} f\big|_{P_0} \cdot \vec{u}\right) ds$.

$$\vec{u} = \frac{2\hat{i} + 2\hat{j} - 2\hat{k}}{\sqrt{2^2 + 2^2 + (-2)^2}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right).$$

$$\frac{\partial f}{\partial x}\Big|_{P_0} = \left(e^x \cos yz\right)\Big|_{(0,0,0)} = e^0 \cos 0 \cdot 0 = 1.$$

$$\frac{\partial f}{\partial y}\Big|_{P_0} = \left(-ze^x \sin yz\right)\Big|_{(0,0,0)} = 0.$$

$$\frac{\partial f}{\partial z}\Big|_{P_0} = \left(-ye^x \sin yz\right)\Big|_{(0,0,0)} = 0.$$

So $\vec{\nabla} f\big|_{P_0} = (1, 0, 0)$.

and $\vec{\nabla} f\big|_{P_0} \cdot \vec{u} = (1, 0, 0) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

$$= \frac{1}{\sqrt{3}}$$

So $df = \left(\vec{\nabla} f\big|_{P_0} \cdot \vec{u}\right) ds$

$$= \frac{1}{\sqrt{3}} \cdot 0.1 = \boxed{\frac{1}{10\sqrt{3}}}.$$

13.6

24. By about how much will

$$h(x, y, z) = \cos(\pi xy) + xz^2$$

change if the point $P(x, y, z)$ moves from $P_0(-1, -1, -1)$ a distance of $ds = 0.1$ unit toward the origin?

Sol'n:

By textbook $dh = \left(\vec{\nabla} h\big|_{P_0} \cdot \vec{u}\right) ds$

$\vec{u} = \dfrac{P_1 - P_0}{\|P_1 - P_0\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$

$\dfrac{\partial h}{\partial x}\Big|_{(-1,-1,1)} = \left(-\pi y \sin(\pi xy) + z^2\right)\Big|_{(-1,-1,-1)} = \pi \sin\pi + (-1)^2 = 1$

$\dfrac{\partial h}{\partial y}\Big|_{(-1,-1,-1)} = \left(-\pi x \sin(\pi xy)\right)\Big|_{(-1,-1,-1)} = -\pi \sin\pi = 0.$

$\dfrac{\partial h}{\partial z}\Big|_{(-1,-1,-1)} = 2xz\Big|_{(-1,-1,-1)} = 2$

So $dh = \left(\vec{\nabla} h\big|_{(-1,-1,-1)} \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right) \cdot 0.1 = \left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right) 0.1 = \boxed{\frac{3}{10\sqrt{3}}}$

13.6

Finding Linearizations

In Exercises 27–32, find the linearization $L(x, y)$ of the function at each point.

32. $f(x, y) = e^{2y-x}$ at a. $(0,0)$, b. $(1,2)$

Sol'n: $L(x,y) = f(P_0) + \vec{\nabla} f(P_0) \cdot ((x,y) - P_0)$

a) $P_0 = (0,0)$. $f(0,0) = e^{2\cdot 0 - 0} = 1$.

$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left.\left(-e^{2y-x}\right)\right|_{(0,0)} = -1$. $\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \left.\left(2e^{2y-x}\right)\right|_{(0,0)} = 2$.

So $L(x,y) = 1 + (-1, 2) \cdot (x, y) = 1 - x + 2y$ //

b) $P_0 = (1,2)$. $f(1,2) = e^{2(2)-1} = e^3$

$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = \left.\left(-e^{2y-x}\right)\right|_{(1,2)} = -e^3$ $\left.\frac{\partial f}{\partial y}\right|_{(1,2)} = \left.\left(2e^{2y-x}\right)\right|_{(1,2)} = 2e^3$.

So $L(x,y) = e^3 + (-e^3, 2e^3) \cdot (x-1, y-2)$

$$= e^3 - e^3(x-1) + 2e^3(y-2)$$

$$= e^3(1 - x + 1 + 2y - 4)$$

$$= e^3(-x + 2y - 2).$$ //.

13.6

Linearizations for Three Variables

Find the linearizations $L(x, y, z)$ of the functions in Exercises 41–46 at the given points.

44. $f(x, y, z) = (\sin xy)/z$ at

a. $(\pi/2, 1, 1)$ **b.** $(2, 0, 1)$

Sol'n: $L(x,y,z) = f(P_0) + \vec{\nabla} f(P_0) \cdot ((x,y,z) - P_0)$.

a) $P_0 = (\frac{\pi}{2}, 1, 1)$. $f(\frac{\pi}{2}, 1, 1) = \frac{\sin(\frac{\pi}{2})}{1} = 1$.

$$\left.\frac{\partial f}{\partial x}\right|_{(\frac{\pi}{2},1,1)} = \left.\left(\frac{y\cos(xy)}{z}\right)\right|_{(\frac{\pi}{2},1,1)} = \frac{1\cdot\cos(\frac{\pi}{2}\cdot 1)}{1} = 0.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(\frac{\pi}{2},1,1)} = \left.\left(\frac{x\cos(xy)}{z}\right)\right|_{(\frac{\pi}{2},1,1)} = \frac{\frac{\pi}{2}\cos(\frac{\pi}{2}\cdot 1)}{1} = 0.$$

$$\left.\frac{\partial f}{\partial z}\right|_{(\frac{\pi}{2},1,1)} = \left.\left(-\frac{\sin(xy)}{z^2}\right)\right|_{(\frac{\pi}{2},1,1)} = \frac{-\sin(\frac{\pi}{2}\cdot 1)}{1^2} = -1$$

So $L(x,y,z) = 1 + (0,0,-1)\cdot\left(x-\frac{\pi}{2}, y-1, z-1\right) = 1-(z-1) = 2-z$

b) $P_0 = (2,0,1)$ $\quad f(2,0,1) = \frac{\sin(2\cdot 0)}{1} = 0.$

$$\left.\frac{\partial f}{\partial x}\right|_{(2,0,1)} = \left.\left(\frac{y\cos(xy)}{z}\right)\right|_{(2,0,1)} = \frac{0\cdot\cos(2\cdot 0)}{1} = 0.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(2,0,1)} = \left.\left(\frac{x\cos(xy)}{z}\right)\right|_{(2,0,1)} = \frac{2\cdot\cos(2\cdot 0)}{1} = 2.$$

$$\left.\frac{\partial f}{\partial z}\right|_{(2,0,1)} = \left.\left(\frac{-\sin(xy)}{z^2}\right)\right|_{(2,0,1)} = \frac{-\sin(2\cdot 0)}{1^2} = 0.$$

So $L(x,y,z) = 0 + (0,2,0)\cdot\left(x-\frac{\pi}{2}, y, z-1\right) = 2y$